INTEGER PROGRAMMING MODELING FOR THE CHINESE POSTMAN PROBLEMS

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ABSTRACT

As far as the traditional Chinese Postman Problem (CPP) is concerned, based on the discussions in the undirected and directed graphs respectively, the corresponding integer programming models are proposed, some numerical examples are given to demonstrate the utility of the models. Furthermore, the models are extended to the case with stochastic weights (the corresponding problem is called Stochastic Chinese Postman Problem). Finally, some possible generalizations of the Chinese Postman Problem are discussed briefly.

Keywords: Chinese Postman Problem, Integer programming, optimal model, weighted graph

INTRODUCTION

The research on optimal delivery route of the postman problem is first proposed and studied by China's professor Guan Mei gu in 1960s, Now known as the Chinese postman problem. Professor Guan Mei gu had given a parity point diagram method for this problem in1960s [1] . In1973 Edmonds and Jonhson gave the improved algorithm of the Chinese postman problem [2] , which is more effective than the former calculation. In 1984 Professor Guan Mei gu reviews the research of the Chinese postman problem before that time in the paper [3] .

The early discussions of the Chinese postman problem is always based on undirected graph, In fact, due to a single line or the slope of the up and down route and so on, The problem sometimes must be resorted to directed graph to study and resolve. So far, on the Chinese postman problem research is mainly carried out from the perspective of graph theory, the majority is given a variety of heuristic algorithms or recursive algorithm, the typical research is [4-12]. This article will research from the perspective of mathematical programming. Mathematic programming model with software package has the advantages of solving convenient, easy to modify and promotion and so on, which even has the advantage of solving large and complex problems.

This paper first establish a corresponding explicit integer programming model on the basis of traditional Chinese postman problem, and the numerical example illustrates the validity of this model, it also pointed out that a slight expansion of this model can also be used to solve the problem based on a directed graph of the generalized Chinese postman problem and the random Chinese postman problem, the modeling and numerical value is also given accordingly.

MODELING OF TRADITIONAL CHINESE POSTMAN PROBLEM

The traditional Chinese postman problem can be summarized as follows:

A postman starts from his post office to go to messenger every time, traveled every street that he or she is responsible for the delivery of the within range, After completion of the messenger task back to the original post office, what kind of routes he or she should choose In order to make the shortest distance to go. The problem abstracted into graph theory problem is that given a connected graph G (V, E), where V = {V1, V2, ... , Vn} is a set of vertex, said that the place of the street intersection, E is the set of edges between the vertices, said street, E = {e = (Vi, Vj);edge e between vertices Vi and Vj}, each edge e in E with non negative weight W (e) = w (Vi, Vj), that the edge is the length of the street, the problem is to determine a circle of G, which over each edge at least once and makes the minimum of the total weight (on each side of the total)of the circle.

From the graph theory, If G does not contain the singularity (the number of adjacent edges is odd), then G has a circle which over each edge once and only once. So this circle is the circle required. If G contains a singularity (the number of adjacent edges is odd), then G is a circle on each side at least once, and will over some side by more than once, If adopted the k-th edge e, let e = (Vi, Vj), we add k-1 new edges between Vi and Vj, and the weight of the new edge is equal to the weight of the edge e, said the new edge is the added edge of edge e. Obviously ,if there are more than one add edge of edge e, we lose an even number of edges, then a total weight obtained from graph of any side over at least once of a circle will not be increased. So we can assume that the number of add edge of each edge add up to one. In this way, problem of finding the optimal delivery routes of the postman can be attributed to the following graph theory problem:

Given a connected graph G(V,E), add an edge $e'=(V_j, V_i)$ between each edge e=(Vi,Vj) ∈ E corresponding to the vertex Vi and Vj, get a connected graph $G'(V,E')$ whose number of edges is double of G , seek $E_1 \subseteq E', E_1 \supseteq E$ in order to make the graph $G_1(V,E_1)$ contain no singularity and the total weight $\sum_{e\in E_1} w(e)$ to a minimum .For simplicity, if $e=(V_i,V_j) \in E'$, it is recorded as $e_{i,j} \in E$ or (i, j), while the corresponding add edge is $e_{j,i}$,and corresponds to edge $e_{i,j} \in E'$,set a 0-1 integer variable $x_{i,j}$,If $e_{i,j} \in E'$,the edge is from Vi to Vj or called arc, In this way, we can put an undirected graph be understood as a directed graph. Every E1 uniquely corresponds to a set of values of $x_{i,j}$, vice versa. We can use the variables $x_{i,j}$ (i=1,2, ... , n; j =1,2, ... , n) to define the constraints of optimal postman problem are as follows:

(1) over each edge at least once and add edge up to at most one, values (known as value system of E1)corresponded to E1 need to meet:

For $\forall e_{i,j} \in E, \quad x_{i,j} + x_{j,i} \ge 1$

(2) graph $G_1(V,E_1)$ contains no singularity, for any vertex Vi, with a "into" the arc, there will be an equivalent amount of" out" to the arc:

$$\sum_{j\neq i} x_{j,i} - \sum_{j\neq i} x_{i,j} = 0$$

The objective of the problem is making the minimum total weight of $G_1(V,E_1)$,that is $\min \sum_{(i,j)\in E'} w_{i,j}x_{i,j}$, $w_{i,j}$ is the weight of edge $e_{i,j}$, $w_{j,i} = w_{i,j}$,so we can get the explicit integer programming model of Chinese postman problem (CPP) as follows:

$$\min \quad \sum_{(i,j)\in E'} w_{i,j}x_{i,j}$$

$$s.t. \quad \begin{cases} \sum\limits_{\substack{j \\ (j,i)\in E'}} x_{j,i} - \sum\limits_{\substack{j \\ (i,j)\in E'}} x_{i,j} = 0, & i = 1,2,...,n \\ x_{i,j} + x_{j,i} \ge 1, & \forall (i,j)\in E \\ x_{i,j} = 0 \text{or} 1, & \forall (i,j)\in E' \end{cases}$$

This model not only can be used to solve the Chinese postman problem, and can also determine the corresponding optimal delivery route: such as $x_{i,j}$ = 1, means that the postman from Vi along the edge (i.e. the street) eij to Vj.

GENERALIZED CHINESE POSTMAN PROBLE AND ITS MODELING

The front section of the postman problem assumes that the postman delivered within the scope of every street in the uplink and downlink without difference, in fact that the delivery of the letter is probably not the case, such as in a case of the street of single line street with a certain slope on both sides of the street cannot be delivered at one time by a single line and so on. This postman problem which we call the generalized Chinese postman problem the generalized Chinese postman problem can be abstracted as a directed graph problem.

Similar to the preceding postman problem (known as traditional Chinese postman problem), generalized postman problem can be described as follows:

Given a connected directed graph G (V, E), each arc e has non-negative weight w (e), we need to find a loop of G which over each arc at least once and make the total weight to a minimum.

For the generalized Chinese postman problem, the number of add arc up to one sometimes is no longer feasible, which needs many add arc to make the corresponding connected to any one of the vertices of a graph G into number of arcs and out number of arcs to the same, in order that the $G'(V,E')$ has a loop (E loop). Here, if $e=(V_i,V_j) \in E$, then we say that arc e is into the arc of vertex Vj, which is also out to the arc of vertex Vi .If the number of vertices of G (V, E) is n, it can be proved that each arc is increased (n-1) add arc at most, each vertex can be realized in the number of in to the arc is equal to the number of out to the arc.

Based on the analysis above, defined a positive integer variable $x_{i,j}$ for each arc $e_{i,j} \in E$ of G (V, E), which is used to represent the arc eij increased ($x_{i,j}$ -1) add arc, thus forming anther directed graph $G'(V, E')$. Similar to the analysis of the previous section, we have the following generalized Chinese postman problem explicit integer programming model (GCPP):

$$\min \sum_{(i,j)\in E} w_{i,\ j} x_{i,\ j}$$

$$s.t. \begin{cases} \sum\limits_{\substack{j \\ (j,i)\in E}} x_{j,i} - \sum\limits_{\substack{j \\ (i,j)\in E}} x_{i,j} = 0, & i = 1, 2, \ldots, n \\ x_{i,\ j} = 1, 2, \ldots, n & \forall \ (i, j) \in E \end{cases}$$

By solving this model, we can get the optimal delivery route of the generalized Chinese postman problem.

Several numerical examples

[Example 1]

Consider the Chinese postman problem as shown below:

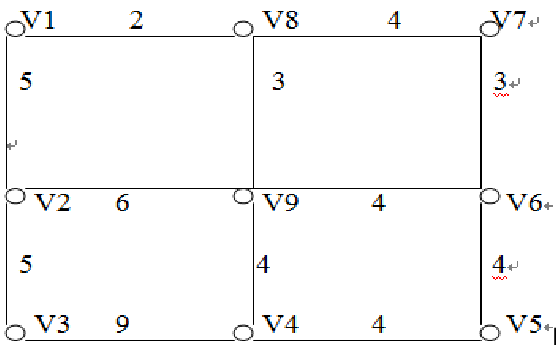

Fig 1 traditional Chinese postman problem (a)

The undirected graph problem is equivalent to the following directed graph:

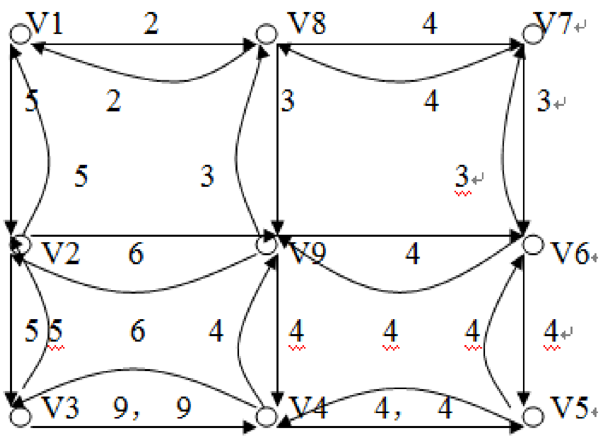

Fig 2 traditional Chinese postman problem (b)

According to the previous discussion of the model, the numerical example shows the corresponding integer programming model as follows:

$$
\min \quad 2(x_{1,8}+x_{8,1})+4(x_{8,7}+x_{7,8})+5(x_{1,2}+x_{2,1})+3(x_{8,9}+x_{9,8})+3(x_{6,7}+x_{7,6})+6(x_{2,9}+x_{9,2})
$$
$$
+4(x_{9,6}+x_{6,9})+5(x_{3,2}+x_{2,3})+4(x_{9,4}+x_{4,9})+4(x_{5,6}+x_{6,5})+9(x_{3,4}+x_{4,3})+4(x_{4,5}+x_{5,4})
$$

$$
s.t.\begin{cases}
x_{2,1}+x_{8,1}-x_{1,2}-x_{1,8}=0\\
x_{1,2}+x_{9,2}+x_{3,2}-x_{2,1}-x_{2,9}-x_{2,3}=0\\
x_{2,3}+x_{4,3}-x_{3,2}-x_{3,4}=0\\
x_{3,4}+x_{9,4}+x_{5,4}-x_{4,3}-x_{4,9}-x_{4,5}=0\\
x_{6,5}+x_{4,5}-x_{5,6}-x_{5,4}=0\\
x_{7,6}+x_{9,6}+x_{5,6}-x_{6,7}-x_{6,9}-x_{6,5}=0\\
x_{6,7}+x_{8,7}-x_{7,6}-x_{7,8}=0\\
x_{1,8}+x_{7,8}+x_{9,8}-x_{8,1}-x_{8,7}-x_{8,9}=0\\
x_{2,9}+x_{4,9}+x_{6,9}+x_{8,9}-x_{9,2}-x_{9,4}-x_{9,6}-x_{9,8}=0\\
x_{1,8}+x_{8,1}\ge 1\\
x_{8,7}+x_{7,8}\ge 1\\
x_{1,2}+x_{2,1}\ge 1\\
x_{8,9}+x_{9,8}\ge 1\\
x_{6,7}+x_{7,6}\ge 1\\
x_{2,9}+x_{9,2}\ge 1\\
x_{9,6}+x_{6,9}\ge 1\\
x_{2,3}+x_{3,2}\ge 1\\
x_{9,4}+x_{4,9}\ge 1\\
x_{5,6}+x_{6,5}\ge 1\\
x_{3,4}+x_{4,3}\ge 1\\
x_{4,5}+x_{5,4}\ge 1\\
x_{i,\ j}=0 \text{or} 1
\end{cases}
$$

The application of integer programming solvers QSB [14] ,and the solution of the optimal solution of the problem is as follows:

$$
x_{1,2}=x_{2,1}=x_{1,8}=x_{8,1}=x_{8,7}=x_{9,8}=
$$
$$
x_{7,6}=x_{2,9}=x_{3,2}=x_{6,9}=x_{4,3}=x_{9,4}=
$$
$$
x_{5,6}=x_{6,5}=x_{5,4}=x_{4,5}=1
$$

other $x_{i,\ j}=0$,the minimum weight is 68. Assumed the post office at the apex of vertex V1, the optimal delivery routes as follows:

$$
e_{1,2}\to e_{2,9}\to e_{9,8}\to e_{8,7}\to e_{7,6}\to e_{6,5}\to e_{5,6}\to e_{6,9}\to
$$
$$
e_{9,4}\to e_{4,5}\to e_{5,4}\to e_{4,3}\to e_{3,2}\to e_{2,1}\to e_{1,8}\to e_{8,1}
$$

We should note that the optimal delivery route of this problem is not unique, similarly we can obtain the optimal delivery route started from any vertex.

[Example 2]

Consider the following are shown in Figure 3 of the generalized Chinese postman problem:

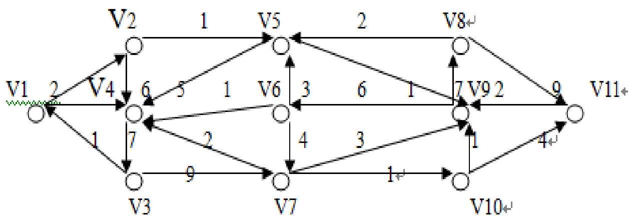

Fig 3 generalized Chinese postman problem

The corresponding integer programming model of generalized Chinese postman problem as follows:

$$\min \quad 2x_{1,2}+8x_{1,4}+x_{3,1}+6x_{2,4}+7x_{4,3}+x_{2,5}+5x_{5,4}+x_{6,4}+2x_{7,2}+9x_{3,7}+4x_{6,7}+3x_{6,5} \\ +2x_{8,5}+x_{5,9}+6x_{9,6}+3x_{7,9}+x_{7,10}+x_{10,9}+7x_{9,8}+9x_{8,11}+2x_{11,9}+4x_{10,11}$$

$$s.t.\begin{cases} x_{3,1}-x_{1,2}-x_{1,4}=0 \\ x_{1,2}-x_{2,4}-x_{2,5}=0 \\ x_{4,3}-x_{3,1}-x_{3,7}=0 \\ x_{1,4}+x_{2,4}+x_{5,4}+x_{6,4}+x_{7,4}-x_{4,3}=0 \\ x_{2,5}+x_{8,5}+x_{6,5}-x_{5,4}-x_{5,9}=0 \\ x_{9,6}-x_{6,5}-x_{6,4}-x_{6,7}=0 \\ x_{6,7}+x_{3,7}-x_{7,4}-x_{7,9}-x_{7,10}=0 \\ x_{9,8}-x_{8,5}-x_{8,11}=0 \\ x_{5,9}+x_{10,9}+x_{11,9}+x_{7,9}-x_{9,6}-x_{9,8}=0 \\ x_{7,10}-x_{10,9}-x_{10,11}=0 \\ x_{10,11}+x_{8,11}-x_{11,9}=0 \\ x_{i,\ j}=1,2,\ldots,10 \end{cases}$$

Application of the same integer programming software, solving this model and obtain the following optimal solution:

$$x_{1,2}=2, x_{1,4}=1, x_{3,1}=3, x_{2,4}=1, x_{4,3}=5, x_{2,5}=1, \\ x_{5,4}=1, x_{6,4}=1, x_{7,4}=1, x_{3,7}=2, x_{6,7}=2, x_{6,5}=1, \\ x_{8,5}=1, x_{5,9}=2, x_{9,6}=4, x_{7,9}=1, x_{7,10}=2, x_{10,9}=1, \\ x_{9,8}=2, x_{8,11}=1, x_{11,9}=2, x_{10,11}=1$$

The minimum weight is 159. Assumed the post office at the apex of vertex V1, the optimal delivery routes as follows:

$$e_{1,2}\to e_{2,4}\to e_{4,3}\to e_{3,1}\to e_{1,4}\to e_{4,3}\to e_{3,1}\to e_{1,2} \\ \to e_{2,5}\to e_{5,4}\to e_{4,3}\to e_{3,7}\to e_{7,10}\to e_{10,11}\to e_{11,9} \\ \to e_{9,8}\to e_{8,11}\to e_{11,9}\to e_{9,6}\to e_{6,5}\to e_{5,9}\to e_{9,8} \\ \to e_{8,5}\to e_{5,9}\to e_{9,6}\to e_{6,7}\to e_{7,9}\to e_{9,6}\to e_{6,4} \\ \to e_{4,3}\to e_{3,7}\to e_{7,10}\to e_{10,9}\to e_{9,6}\to e_{6,7}\to e_{7,4} \\ \to e_{4,3}\to e_{3,1}$$

It is a loop, the optimal delivery routes can be determined of the post office in any vertex similarly. This optimal delivery routes found out is based on the model of the optimal solution with the method similar to the "solitaire" game. It should be noted that if the value of $x_{i,\ j}$ is greater than 1, means that the arc to be repeated the walk.

The sum of all $x_{i,\ j}$ should be equal to the number of street walked (including the walk repeated).

RANDOM CHINESE POSTMAN PROBLEM AND ITS MODELING

Traditional and generalized Chinese postman problem discussed above assume that the weights related to edge or arc are determined constants. It is often encountered that the weight is non-fixed in practice, for example, the weight considered is the delivery time spent on the streets, this parameter is often not constant, the time it takes for each delivery will change with the number of mail, but it generally follows a form of change, that is, the weights are random variables with a certain distribution, then we call the corresponding problem for the random Chinese postman problem.

The parity graphic working method of traditional Chinese postman problem [1] and its improved algorithm [2] cannot solve the random problem, but by means of the integer programming model established in this paper, and with the application of stochastic programming theory [16-18] , the problem can be solved easily.

A variety of approach related to solve random problem, such as expected value method chance constrained method the optimal value distribution method relevant chance constrained method, multi-stage(such as two-stage)method. This paper only discusses the chance constrained method, and its core is the deterministic equivalent treatment process under the constraint conditions of probability. Readers interested in other method may follow the similar treatment process of the stochastic programming theory, which is omitted in the paper.

Noting that the constraint of (CPP) or (GCPP) does not contain a parameter of weight, therefore solving a question of random weight, what need to do is just making the objective function for the corresponding deterministic equivalent treatment process. When weight is a random variable, the objective function is also a random variable, according to theory of stochastic programming, stochastic CPP problem can be transformed into the two deterministic equivalent model as following:

$$1-CPP(\alpha) \quad \min \quad w$$
$$s.t. \quad \begin{cases} P(\sum_{(i,j)\in E'} w_{i,j}x_{i,j} \le w) \ge \alpha \\ \text{constraints of (CPP)} \end{cases}$$

$$2-CPP(w) \quad \max \quad \alpha$$
$$s.t. \quad \begin{cases} P(\sum_{(i,j)\in E'} w_{i,j}x_{i,j} \le w) \ge \alpha \\ \text{constraints of (CPP)} \end{cases}$$

The 1-CPP（α）refers to the solving of $x_{i,j}$ and w when α is fixed, the 2-CPP（w）refers to the solving of $x_{i,j}$ and α when w is fixed. Here, we refer to α as the feasible weights of w, w is the level hoped of total weight. If the weights are subject to normal distribution and independent of each other, $w_{i,j}$ $follows N(\mu_{i,j}, \sigma_{i,j}^2)$,1-CPP（α）and 2-CPP（w）are equivalent to the following two mathematical programming respectively:

$$1-\mathrm{N}CPP(\alpha) \quad \min \quad w$$
$$s.t. \quad \begin{cases} w-\sum_{(i,j)\in E'} \mu_{i,j}x_{i,j} - F^{-1}(\alpha)\sqrt{\sum_{(i,j)\in E'} \sigma_{i,j}^2 x_{i,j}^2} \ge 0 \\ \text{constraints of (CPP)} \end{cases}$$

$$2-\mathrm{N}CPP(w) \quad \max \quad \alpha$$
$$s.t. \quad \begin{cases} w-\sum_{(i,j)\in E'} \mu_{i,j}x_{i,j} - F^{-1}(\alpha)\sqrt{\sum_{(i,j)\in E'} \sigma_{i,j}^2 x_{i,j}^2} \ge 0 \\ \text{constraints of (CPP)} \end{cases}$$

Where F(X) is the standard normal cumulative distribution function, F^{-1}（y）is its inverse function. Further, it is equivalent to the following two mathematical programming problems respectively:

$$1-\mathrm{N}CPP(\alpha)' \quad \min \quad \sum_{(i,j)\in E'} \mu_{i,j}x_{i,j} + F^{-1}(\alpha)\sqrt{\sum_{(i,j)\in E'} \sigma_{i,j}^2 x_{i,j}^2} = w$$
$$s.t. \quad \text{constraints of (CPP)}$$

$$2-\mathrm{N}CPP(w)' \quad \max \quad \frac{w-\sum_{(i,j)\in E'} \mu_{i,j}x_{i,j}}{\sqrt{\sum_{(i,j)\in E'} \sigma_{i,j}^2 x_{i,j}^2}} = F^{-1}(\alpha)$$
$$s.t. \quad \text{constraints of (CPP)}$$

It should be noted that $F^{-1}(\alpha)$ is a monotonically increasing function of α .Other types of random problem can also be analyzed similarly. As the programming above is non-linear, we need to use the non-linear integer programming tools and software or two stage method [18] to solve the problem; large problem can also be solved by means of modern intelligent optimization algorithm [19] .

The corresponding random problem of the generalized Chinese postman problem can also be discussed similarly, which is omitted here, the interested reader may have a try.

POSSIBLE PROMOTION

The paper based on traditional Chinese postman problem established the explicit integer programming model of the problem, and it has been extended to the generalized Chinese postman problem based on the weighted directed graph and random Chinese postman problem with uncertain weights. Other possible promotion is the Chinese postman problem considered the weight of the fuzzy type or grey type or coarse type or belief function type, what we need to do is making a corresponding deterministic conversion of the objective function based on the uncertainty connotation. This is the charm of the mathematical model established by the paper, interested readers can have a try.

This paper discusses the problem of single objective, considering the multi-objective traveling salesman problem, the flexible and mobile and easy to modify and deformation characteristics of mathematical programming can be used to solve the problem of possible expansion easily.

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